Wigner's legacy and the fundamental theorem of Projective Geometry

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Part of diploma thesis at the
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Hamburg, 14 February 2007

Overview

Motivation
Wigner's theorem

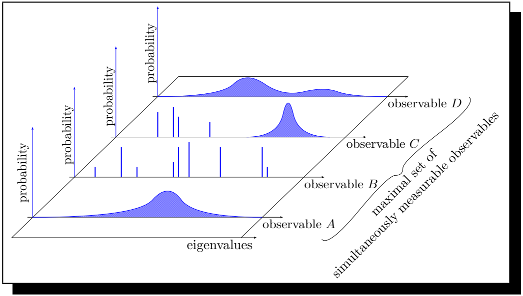
Theorem (Wigner's Theorem)

Any symmetry transformation between quantum states may be represented on Hilbert space either by a unitary or by an anti-unitary transformation.

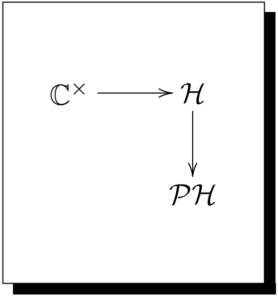
- Wigner's theorem tells us how to treat symmetries in quantum mechanics.
- There is a connection to projective geometry, that was realised decades ago [Uhlhorn(1962), Lomont and Mendelson(1963)], but only partly used to prove Wigner's theorem.

Motivation
States in Hilbert space

Eigenvalues and probabilities do not depend on length or phase of vectors in Hilbert space $\mathcal{H}$.



Quantum mechanical state



Principal bundle

Projective Geometry
Basic notation

Definitions

- A *ray* A is an orbit of the group of unities $\mathbb{K}^\times$ on a vector space V over the field $\mathbb{K}$. We call $a \in \mathrm{A}$ a *representative* of A, and write $[a] = \mathrm{A}$.
- The set of all rays in V is called *projective space* $\mathcal{P}V$.
    - $\dim \mathcal{P}V = \dim V - 1$
- There is a natural operation on $\mathcal{P}V$, the *unification*:

$$\mathrm{A} \vee \mathrm{B} := \{a + b \in V : \, a \in \mathrm{A}, \, b \in \mathrm{B}\} \, .$$

- n points $\mathrm{A}_1, \ldots, \mathrm{A}_n \in \mathcal{P}V$ are called *projectively independent*, iff they span a projective subspace of dimension $n-1$, i.e.

$$\exists a_i \in \mathrm{A}_i : \, \{a_1, \ldots, a_n\} \subset V \text{ is linearly independent.}$$

- Three points $\mathrm{A}, \mathrm{B}, \mathrm{C} \in \mathcal{P}V$ are called *collinear*, iff $\mathrm{C} \in \mathrm{A} \vee \mathrm{B}$.

Projective Geometry
Maps between projective spaces

Definitions

- A bijective map $\mathbf{K} : \mathcal{P}V \to \mathcal{P}W$ is called *a collineation*, iff $\mathbf{K}$ respects collinearity, i.e. $\forall\, \mathrm{A}, \mathrm{B} \in \mathcal{P}V$:
$$\mathbf{K}\,(\mathrm{A} \vee \mathrm{B}) = \mathbf{K}\mathrm{A} \vee \mathbf{K}\mathrm{B}\,.$$
- A bijective map $\mathbf{S} : \mathcal{P}V \to \mathcal{P}W$ is called a *semi-projectivity*, iff there is a *compatible* semi-linear map $S : V \to W$, i.e. $\forall\, \mathrm{A} \in \mathcal{P}V$:
$$[Sa] = \mathbf{S}\,[a]\ .$$

Theorem (main theorem of projective geometry)

Let $\mathcal{P}V$ and $\mathcal{P}W$ be projective spaces of (finite) dimension $n \geq 2$ and let $\mathbf{K} : \mathcal{P}V \to \mathcal{P}W$ be a collinearity. Then $\mathbf{K}$ is a semi-projectivity.

Symmetries

Projective Hilbert space - the space of states

- Take as vector space some Hilbert space $(\mathcal{H}, \langle\cdot|\cdot\rangle)$ over the field $\mathbb{C}$.
- Then there is an additional operation $\odot : \mathcal{PH} \times \mathcal{PH} \to [0,1]$, induced by the scalar product $\langle\cdot|\cdot\rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$:

$$\mathrm{A} \odot \mathrm{B} := \frac{|\langle a|b\rangle|^2}{\|a\|^2 \, \|b\|^2},$$

which only depends on the rays, not on the vectors in $\mathcal{H}$ and is of course the *transition probability* between the states A and B.

Definition (projective Hilbert space)

A *projective Hilbert space* is a pair $(\mathcal{PH}, \odot)$, where $\mathcal{PH}$ is a projective space and $\odot : \mathcal{PH} \times \mathcal{PH} \to [0,1]$ is the map induced by the scalar product on $\mathcal{H}$ as shown above.

Symmetries

Symmetry transformation and Wigner's theorem

Definition (Symmetry transformation)

A *symmetry transformation* on states is a map $\mathbf{T} : \mathcal{PH} \to \mathcal{PH}$, that respects transition probabilities, i.e. $\forall\, \mathrm{A}, \mathrm{B} \in \mathcal{PH}$:

$$\mathbf{T}\mathrm{A} \odot \mathbf{T}\mathrm{B} = \mathrm{A} \odot \mathrm{B}\,.$$

We call this property of $\mathbf{T}$ *quasi-unitarity*.

Theorem (Wigner's theorem - revisited)

There exists a map U such that this diagram commutes:

$$\begin{array}{ccc} \mathcal{H} & \overset{U}{\dashrightarrow} & \mathcal{H} \\ \downarrow & & \downarrow \\ \mathcal{PH} & \xrightarrow{\mathbf{T}} & \mathcal{PH} \end{array}$$

U: *semi-unitary*

$\mathbf{T}$: *quasi-unitary*

Proof of Wigner's theorem
Finite dimensional Hilbert space

Proposition (quasi-unitarity → collinearity)

Any quasi-unitary transformation on projective Hilbert space is a collineation.

Theorem (main theorem of projective geometry)

Any collineation is a semi-projectivity.

Conclusion (linearity → unitarity)

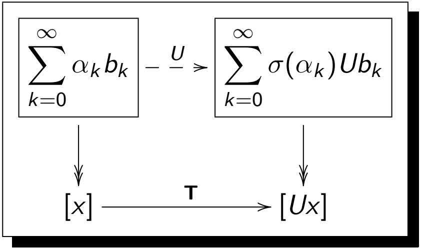
Any quasi-unitary and thereby semi-projective transformation $\mathbf{T}$ on projective Hilbert space $\mathcal{PH}$ admits a compatible semi-unitary transformation U on Hilbert space $\mathcal{H}$, i.e

$$\exists U : \mathcal{H} \to \mathcal{H} \quad \text{such that} \quad [Ua] = \mathbf{T}\,[a]\ .$$

Proof of Wigner's theorem

Infinite dimensions

Final step is to prove Wigner's theorem also for infinite sums:



$U : \mathcal{H} \to \mathcal{H}$ - compatible to symmetry $\mathbf{T}$, i.e. $\mathbf{T}[x] = [Ux]$.
- maps ONB $\{b_k\}_{k \in I}$ to ONB $\{Ub_k\}_{k \in I}$.

$\sigma : \mathbb{C} \to \mathbb{C}$ field automorphism, i.e. $\sigma \in \{\mathrm{id}, \overline{\mathrm{id}}\}$; $\alpha_k \in \mathbb{C}$.

Summary

- There is a projective bundle underlying the notion of a state in quantum mechanics.
- The corresponding methods of projective geometry lead to a geometric proof of Wigner's theorem.
- Wigner's theorem is nothing else but a natural generalization of the fundamental theorem of projective geometry towards projective Hilbert space.

MORE ON THE TOPIC

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Arkiv för Fysik, 23(30):307–340, 1962.

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The wigner unitary-antiunitary theorem.
Ann. Math., 78(3):548–559, 1963.

V. Bargmann.
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This talk and further details:

`wwwthep.physik.uni-mainz.de/~keller`