Quantum indistinguishability and rotation generators

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- Noether's Theorem (1918, Emmy Noether)

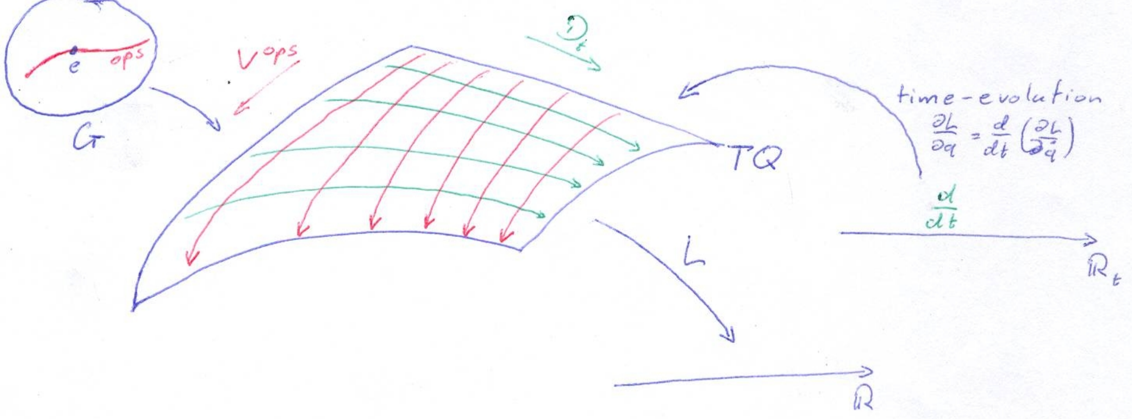
• given a physical system $(TQ, L: TQ \to \mathbb{R})$
Lagrangian

→ time-evolution $\left(ELG: \frac{\partial L}{\partial q} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right)\right)$
induces flux and associated VF on TQ $\quad D_t \in VF(TQ)$

• given a group-action on Q $(\to TQ)$

$$G \to \mathrm{Diff}(TQ)$$

→ one-parameter-subgroup of G
induces flux and associated VF on TQ $\quad V^{ops} \in VF(TQ)$



⇒ Noether's Theorem combines both fluxes by:

$$\boxed{\mathcal{L}_{V^{ops}} L = D_t (p\, V^{ops})} \qquad p = \frac{\partial L}{\partial \dot q^i} dq^i \in \Omega^1(TQ)$$

$\Rightarrow$ conserved quantities if $\mathcal{L}_{V^{ops}} L = 0$ obvious.

But if $\mathcal{L}_{V^{ops}} L \neq 0$, we hope for $\mathcal{L}_{V^{ops}} L = \frac{d}{dt} K$,

then $\frac{d}{dt}(\rho V^{ops} - K) = 0$

Exactly this is the case if we regard the magnetic monopole

MAGNETIC MONOPOLE

NB: "toy-model" space of m.m. is $\mathbb{R}^3 \setminus \{0\}$ which is of same homotopy class as S^2.
$S^2 \longrightarrow \mathbb{R}P^2$ is covering-space. $\mathbb{R}P^2$ is interesting part of configuration space of two indistinguishable particles

$$L(q,\dot q) = \frac{1}{2} m \dot q^2 + \frac{e}{c} \dot q \cdot A(q)$$

regard potential $A^N : \mathbb{R}^3 \setminus \{\text{negative } z\text{-Axis}\} \longrightarrow \mathbb{R}$

$$A^N(q) = \frac{g}{r(r+q_3)} (-q_2, q_1, 0)$$

Take as group: $SO(3)$

① ops: rotation round 1-axis (infinitesimally)

$$\begin{aligned} q_1 &\longmapsto q_1 \\ q_2 &\longmapsto q_2 + \varepsilon q_3 \\ q_3 &\longmapsto q_3 - \varepsilon q_2 \end{aligned} \quad \Rightarrow V_1^{ops} = (0, +q_3, -q_2, 0, +\dot q_3, -\dot q_2)$$

compute:

$$\left(\mathcal{L}_{V_1^{ops}} L\right)(q,\dot q) = \frac{d}{d\varepsilon}\left(L(R_\varepsilon q, R_\varepsilon \dot q) - L(q,\dot q)\right)\Big|_{\varepsilon=0}$$

$\Rightarrow$ only first order ε^1 counts!

$\Rightarrow \dot q^2$ is rotation-invariant

$$\Rightarrow \text{compute} \frac{d}{d\varepsilon}\left[\frac{1}{g}\underbrace{(\dot q \cdot A)}_{\in F(TQ)}(R_\varepsilon(q), R_\varepsilon(\dot q)) - \frac{1}{g}\underbrace{(\dot q \cdot A)(q,\dot q)}_{\in F(TQ)}\right]_{\varepsilon=0}$$

$$\Rightarrow \boxed{\mathcal{L}_{V_1^{ops}} L = -\frac{ge}{c}\frac{d}{dt} K_1}$$

② $\mathcal{L}_{V_2^{ops}} L = -\frac{ge}{c}\frac{d}{dt} K_2$

③ $\mathcal{L}_{V_3^{ops}} L = -\frac{ge}{c}\frac{d}{dt} K_3$

$$\Rightarrow \vec K = \boxed{\frac{q + r\hat e_3}{r + q_3}}$$

On the other hand:

$$p V_1^{ops} = p_2 q_3 - p_3 q_2 = (p \times q)_1$$

$$p V_2^{ops} = \quad \cdots \quad = (p \times q)_2$$

$$p V_3^{ops} = \quad \cdots \quad = (p \times q)_3$$

Noether:

$$\frac{d}{dt}\left(p V_i^{ops}\right) = \frac{d}{dt}\left(-\frac{eg}{c} K_i\right)$$

$$\Rightarrow \boxed{\frac{d}{dt}\left(q \times p - \frac{eg}{c}\vec{K}\right) = 0 =: \frac{d}{dt}\left(\vec{J}\right)} ,$$

additionally: $\boxed{\{J_k, J_l\} = \varepsilon_{klm} J_m}$

$\Rightarrow$ Non-trivial configuration-space gives rise to new rotation-generators (Angular momentum)

$$\boxed{\vec{J} = q \times p - \frac{eg}{c} \frac{q + r\hat{e}_3}{r + q_3} = \text{const.}}$$