The Hopf Algebra of Feynman Graphs

An Introduction to Connes-Kreimer Theory of Renormalization

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Overview

Introduction
BPHZ Renormalization - A Short Review

- The basic quantity in perturbative QFT is the vertex function

$$U(\Gamma)(p_1, \ldots, p_N) = \int I_\Gamma(k_1, \ldots, k_L; p_1, \ldots, p_N)\, dk_1 \cdots dk_L \,.$$

- In general the integrals are divergent.
- If $U(\Gamma)$ is only *overall divergent*, we get sensible information by just subtracting divergent parts:

$$R(\Gamma) = U(\Gamma) - T \circ U(\Gamma)\,.$$

$C(\Gamma) := -T \circ U(\Gamma)$ is called the (BPHZ-) counter term.

- **B**ogoliubov, **P**arasiuk, **H**epp and **Z**immermann (1957-1969) developed a general method to construct counter terms for arbitrary 1PI graphs.

Introduction

BPHZ Renormalization - A Short Review

- Replace the original expression $U(\Gamma)$ by

$$\overline{R}(\Gamma) = U(\Gamma) + \sum_{\gamma \subsetneq \Gamma} C(\gamma) U(\Gamma/\gamma)$$

- The counter term $C(\Gamma)$ of an arbitrary graph is then given by

$$C(\Gamma) = -T \circ \overline{R}(\Gamma) = -T\left(U(\Gamma) + \sum_{\gamma \subsetneq \Gamma} C(\gamma) U(\Gamma/\gamma) \right)$$

(it is multiplicative for non-connected subgraphs $\gamma \subsetneq \Gamma$)

- The renormalized value is the sum of the above expressions:

$$\boxed{R(\Gamma) = U(\Gamma) + C(\Gamma) + \sum_{\gamma \subsetneq \Gamma} C(\gamma) U(\Gamma/\gamma)}$$

BPHZ

Introduction
Hopf Algebra - Revision and Fixing Notation

Definition (Hopfalgebra)

A Hopf algebra $(H, m, e, \triangle, \varepsilon, S)$ over a field $\mathbb{K}$ is a bialgebra with antipode, i.e.

- (H, m, e) is an algebra with
$$\begin{array}{rccll} m: & H\otimes H & \longrightarrow & H & \text{(associative multiplication)} \\ e: & \mathbb{K} & \longrightarrow & H & \text{(unit)} \end{array}$$
- There are algebra homomorphisms:
$$\begin{array}{rccll} \triangle: & H & \longrightarrow & H\otimes H & \text{(coassociative comultiplication)} \\ \varepsilon: & H & \longrightarrow & \mathbb{K} & \text{(counit)} \end{array}$$
- There is a map
$$\begin{array}{rcccl} S: & H & \longrightarrow & H & \text{(antipode)} \ , \end{array}$$
which fulfills:
$$S \star \mathrm{id}_H = \mathrm{id}_H \star S = e \circ \varepsilon \, .$$

The Hopf Algebra of Feynman Graphs

Discrete Part

- $J = \{\text{monomials in the Lagrangian } \mathcal{L}_{\mathcal{T}}\}$
- $\mathrm{Graph}(\mathcal{T}) = \left\{\text{extended graphs } \Gamma \text{ with map } \iota : \Gamma^{(0)} \to J\right\}$
- $\gamma = \prod (\tilde{\gamma}_i, \chi(\gamma_i))$ subgraph, $\chi : \{\text{conn. comp. of } \gamma\} \to J$

Definition (discrete Hopf algebra of FG $\mathcal{H}(\mathcal{T})$)

- free, commutative algebra over $\mathbb{C}$
- generated by pairs (Γ, w) with $\Gamma \in \mathrm{Graph}(\mathcal{T})$ a 1PI graph, and $w \in J$ a monomial with $\deg(w) = \#$ ext. lines of Γ.
- The coproduct is defined on the generators as

$$\Delta((\Gamma, w)) = (\Gamma, w) \otimes 1 + 1 \otimes (\Gamma, w) + \sum_{\gamma \subsetneq \Gamma} \gamma \otimes (\Gamma/\gamma, w) \ .$$

To prove: *coassociativity* and *existence of antipode*

The Hopf Algebra of Feynman Graphs

Discrete Part - Grading

- $\mathcal{H}(\mathcal{T})$ is $\mathbb{N}$-graded as a bialgebra, i.e. $\mathcal{H}(\mathcal{T}) = \bigoplus_{n\in\mathbb{N}\cup\{0\}} \mathcal{H}^n$.
- we consider
  $n = b_1(\Gamma) =$ loop number $L(\Gamma) = \#\Gamma^{(1)}_{\text{int}} - \#\Gamma^{(0)} - 1.$
- With respect to this grading $\mathcal{H}(\mathcal{T})$ is $\mathbb{N}$-*graded connected*, i.e. $\mathcal{H}^0 = \mathbb{C}1$.

Fact

Any $\mathbb{N}$-graded connected bialgebra $(H, m, e, \Delta, \varepsilon)$ is a Hopf algebra. The antipode is given by

$$S((\Gamma, w)) = -(\Gamma, w) - \sum_{\gamma\subsetneq\Gamma} S(\gamma)\,(\Gamma/\gamma, w)$$

The Hopf Algebra of Feynman Graphs
Reproducing BPHZ

- $U : \mathcal{H}(\mathcal{T}) \to \mathcal{R}$ is a character; $\mathcal{R}$: ring (of Laurent series).
- The Hopf algebra structure is represented by the character $S_T^U := -T \circ U \circ S \; : \mathcal{H}(\mathcal{T}) \to \mathcal{R}$,

$$S_T^U(\Gamma) = -T \circ \left[U(\Gamma) + \sum_{\gamma \subsetneq \Gamma} S_T^U(\gamma) U(\Gamma/\gamma) \right] .$$

Assertion

- *The map $S_T^U : \mathcal{H}(\mathcal{T}) \to \mathcal{R}$ gives the BPHZ-counterterm.*
- *The renormalized value $R(\Gamma)$ for $\Gamma \in \mathcal{H}(\mathcal{T})$ is given by*

$$R(\Gamma) = \left(S_T^U \star U\right)(\Gamma) .$$

BPHZ

The Hopf Algebra of Feynman Graphs
BPHZ-Counterterms and External Structure of Graphs

- The formalism giving the map T used in CK-theory is

  dimensional regularization and *minimal subtraction*.

- The pole part $C(\text{-○-})$ of $U(\text{-○-})(k)$ in massive ϕ^3-theory has the form

$$C(\text{-○-}) = c_0 m^2 + c_1 k^2$$

- Extracting the different contributions can be done by applying certain distributions to $C(\text{-○-})$ (or directly to $U(\text{-○-})(k)$) giving the coefficients c_i of the above polynomial:

$$\sigma_0\left(c_0 m^2 + c_1 k^2\right) = c_0 \qquad \text{and} \qquad \sigma_1\left(c_0 m^2 + c_1 k^2\right) = c_1 \,.$$

- Possible choice: $\sigma_0(f) := f(k)\big|_{k=0}$ and $\sigma_1(f) = \frac{\partial^2}{\partial k^2} f(k)\big|_{k=0}$.

The Hopf Algebra of Feynman Graphs

Full Structure

- As a function of the external momenta, $U(\Gamma)$ is defined on the space

$$E_\Gamma := \left\{ (p_1, \dots, p_N) : \sum_{i=1}^{N} p_i = 0 \right\} .$$

- For every $\Gamma \in \mathcal{H}(\mathcal{T})$ we choose a distribution

$$\sigma_w : C^\infty(E_\Gamma) \to \mathbb{C} ,$$

  labeled by the monomials $w \in J$ in the Lagrangian.

- The distribution σ_w belonging to a graph Γ is called *external structure* of Γ.

The Hopf Algebra of Feynman Graphs
Full Structure

- $C_c^{-\infty}(E_\Gamma)$: space of (comp. supp.) distributions on E_Γ.
- Consider $E := \bigcup_\Gamma E_\Gamma$, then: $C_c^{-\infty}(E) = \bigoplus_\Gamma C_c^{-\infty}(E_\Gamma)$.

Definition (Hopf algebra of Feynman graphs $\widetilde{\mathcal{H}}(\mathcal{T})$)

- $\widetilde{\mathcal{H}}(\mathcal{T})$ is the symmetric algebra on the linear space of distributions $C_c^{-\infty}(E)$:

$$\widetilde{\mathcal{H}}(\mathcal{T}) = \mathrm{Sym}(C_c^{-\infty}(E)) = \bigoplus_{n\in\mathbb{N}\cup\{0\}} \left(C_c^{-\infty}(E)\right)^{\vee n}.$$

- The generators being pairs (Γ, σ_w) with Γ a 1PI graph of $\mathcal{T}$ and $\sigma_w \in C_c^{-\infty}(E_\Gamma)$.
- The coproduct defined as above:

$$\Delta\left(\Gamma, \sigma_w\right) = \left(\Gamma, \sigma_w\right) \otimes 1 + 1 \otimes \left(\Gamma, \sigma_w\right) + \sum_{\gamma \subsetneq \Gamma} \left(\gamma, \sigma_{\chi(\gamma)}\right) \otimes \left(\Gamma/\gamma, \sigma_w\right)$$

Outlook

- Compatibility of Gauge identities (Ward-, Slavnov-Taylor-) with renormalization (van Suijlekom, 2007).
- Renormalization of multiple zeta values (L. Guo and B. Zhang, 2006)
- Description of Renormalization group, using algebraic geometry.

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