Euclidean Epstein-Glaser Renormalization

Kai J. Keller

II. Institute for Theoretical Physics, Hamburg University

- Part of PhD thesis under supervision of K. Fredenhagen -

Göttingen, 31 January 2009

Motivation

- perturbative Quantum field theory (pQFT):

$$S(V) = \exp_T(V) = \sum_{n=0}^{\infty} \frac{1}{n!} V \cdot_T \cdots \cdot_T V = \sum_{n=0}^{\infty} \frac{1}{n!} S^{(n)}(V^{\otimes n})\,.$$

$\overset{\text{LSZ}}{\Rightarrow}$ Transition probabilities in collisions of elementary particles.

- BUT: $\mathrm{supp}(F) \cap \mathrm{supp}(G) \neq \emptyset \Rightarrow F \cdot_T G$ ill-defined.
- Epstein-Glaser renormalization: One way to give sense to $V \cdot_T V$ (at least for local interaction functionals V).
- Other renormalization schemes like *BPHZ* or *DimReg+MS*, are often performed in spaces of Euclidean signature.
- Motivation for Euclidean EG: Connection to other schemes.

Overview

The Wave Front Set

$u : \mathscr{D}(\mathbb{R}^n) \to \mathbb{C}$: distribution.

Singular Support

$$\mathrm{singsupp}(u) := \left\{ x \in \mathbb{R}^n : \nexists \mathcal{U}_x \ni x : \left. u \right|_{\mathcal{U}_x} \in C^\infty(\mathcal{U}_x) \right\}$$

Wave Front Set

$$\mathrm{WF}(u) = \left\{ (x,k) \in \dot{T}^*\mathbb{R}^n : x \in \mathrm{singsupp}(u), \begin{array}{l} \forall f : \text{ Fourier transform } \widehat{fu} \text{ does} \\ \textit{not} \text{ decay rapidly in direction } k \end{array} \right\}$$

Microlocal Analysis $u, v \in \mathscr{E}'(\mathbb{R}^n)$, D: differential operator

$$\mathrm{WF}(Du) \subset \mathrm{WF}(u) \subset \mathrm{WF}(Du) \cup \mathrm{Char}(D)$$

$$0 \notin \mathrm{WF}(u) \oplus \mathrm{WF}(v) \qquad \Rightarrow \qquad \exists!\, u \cdot v \in \mathscr{E}'(\mathbb{R}^n)\,.$$

pAQFT: The Algebra of Observables

Quantum Product [Brunetti, Dütsch, Fredenhagen, ... 1996-2009]

$$\mathcal{F}(\mathbb{M}) := \left\{ F : \mathscr{E}(\mathbb{M}) \to \mathbb{C} : F^{(n)}(\varphi) \in \mathscr{E}'(\mathbb{M}^n), \left[\mathrm{WF}(F^{(n)}(\varphi)) \right]_2 \cap \left(\overline{V_-}^n \cup \overline{V_+}^n \right) = \emptyset \right\}$$

- contains field monomials: $F(\varphi) = \frac{1}{k!} \int dx\, (\varphi(x))^k f(x)$, $f \in \mathscr{D}(\mathbb{M})$
- local interactions are a subset $\mathcal{F}_{\mathrm{loc}}(\mathbb{M}) \subset \mathcal{F}(\mathbb{M})$.

Quantum product on $\mathcal{F}(\mathbb{M})[[\hbar]]$

$$F \star_\hbar G = \sum_{k=0}^{\infty} \frac{\hbar^k}{k!} \left\langle F^{(n)}, \Delta_+^{\otimes n} G^{(n)} \right\rangle$$

Δ_+: positive frequency part of causal propagator $\Delta = \Delta_{\mathrm{ret}} - \Delta_{\mathrm{adv}}$.

$$\boxed{\left(\Box + m^2 \right) \Delta_+ = 0}$$

- $F \star_\hbar G$ well-defined (as formal power series in $\hbar$).

pAQFT: The Time Ordered Product
Renormalization Problem

Time Ordering Operator

$$T := \exp\left(\hbar \Gamma_{\Delta_F}\right), \quad \Gamma_{\Delta_F} := \frac{1}{2}\int dx\, dy\, \Delta_F(x-y)\, \frac{\delta^2}{\delta\varphi(x)\,\delta\varphi(y)}$$

induced product:

$$\begin{array}{ccc} \mathcal{F}(\mathbb{M})[[\hbar]]^{\otimes 2} & \xrightarrow{T^{\otimes 2}} & \mathcal{F}(\mathbb{M})[[\hbar]]^{\otimes 2} \\ M \downarrow & \# & \downarrow \cdot_T \\ \mathcal{F}(\mathbb{M})[[\hbar]] & \xrightarrow{T} & \mathcal{F}(\mathbb{M})[[\hbar]] \end{array}$$

$$F \cdot_T G = \sum_{n=0}^{\infty} \frac{\hbar^n}{n!}\left\langle F^{(n)}, \Delta_F^{\otimes n} G^{(n)}\right\rangle$$

- $\boxed{\left(\Box + m^2\right)\Delta_F = i\delta} \quad \Rightarrow \quad \mathrm{WF}(\delta) \subset \mathrm{WF}(\Delta_F)$
- $F \cdot_T G$ ill-defined if $\mathrm{supp}(F) \cap \mathrm{supp}(G) \neq \emptyset$

pAQFT: Epstein-Glaser "in a nutshell"

Causality $\mathrm{supp}(F)$ "later than" $\mathrm{supp}(G) \Rightarrow F \cdot_T G = F \star_\hbar G\,.$

Epstein-Glaser-Induction $F_1, \dots, F_n \in \mathcal{F}_{\mathrm{loc}}(\mathbb{M})[[\hbar]]$,
then $S^{(n)}(F_1 \otimes \cdots F_n)$ can be defined up to $\mathrm{Diag}(\mathbb{M}^n)$.

Scaling degree Let $t \in \mathscr{D}'(\mathbb{R}^d)$, then

$$\mathrm{sd}(t) := \inf \left\{ \omega \in \mathbb{R} : \lim_{\lambda \searrow 0} \lambda^\omega t_\lambda = 0 \right\}, \quad t_\lambda(f) = \int_{\mathbb{R}^d} t(\lambda x)\, f(x)\, dx$$

Theorem (Brunetti, Fredenhagen 2000)

$t_0 \in \mathscr{D}'(\mathbb{R}^d \setminus \{0\})$ *with scaling degree* $\mathrm{sd}(t_0)$ *w.r.t. the origin,*

- $\mathrm{sd}(t_0) < d \Rightarrow \exists!$ *extension* $t \in \mathscr{D}'(\mathbb{R}^d)$ *of* t_0*:* $\mathrm{sd}(t) = \mathrm{sd}(t_0)$.
- $\mathrm{sd}(t_0) \geq d \Rightarrow \exists$ *extensions* $t \in \mathscr{D}'(\mathbb{R}^d)$ *of* t_0*:* $\mathrm{sd}(t) = \mathrm{sd}(t_0)$, *uniquely defined by values on a finite set of test functions.*

Euclidean Framework

What is gained / lost?

"Wick rotation" $(x_0, x_1, x_2, x_3) \mapsto (e_1, e_2, e_3, e_4) = (ix_0, x_1, x_2, x_3)$,

$$x_0^2 - x_1^2 - x_2^2 - x_3^2 \mapsto -\left(e_1^2 + e_2^2 + e_3^2 + e_4^2\right)$$
$$\Box + m^2 \mapsto -\Delta + m^2$$

⊕ Calculations become easier: absolutely convergent integrals, ...

⊖ Causal structure is completely lost

- Helmholtz operator $\left(-\Delta + m^2\right)$ is elliptic, hence it has a unique fundamental solution P:

$$\left(-\Delta + m^2\right) P = \delta \qquad \Rightarrow \qquad \mathrm{WF}(P) = \mathrm{WF}(\delta)$$

$\Rightarrow$ Euclidean Epstein-Glaser recursion needs to be performed with only one fundamental solution, i.e. one product, which is generally ill-defined.

Partial Algebra of Functionals

Euclidean Time Ordered Product

$$\mathcal{F}(\mathbb{E}) = \left\{ F : \mathscr{E}(\mathbb{E}) \to \mathbb{C} : \ \forall n \in \mathbb{N} : \ F^{(n)}(\varphi) \in \mathscr{E}'(\mathbb{E}^n) \right\}$$

Euclidean T_E-Operator

$$T_E = \exp(\hbar \Gamma_P)\,, \qquad \Gamma_P = \int dx\, dy\, P(x-y)\, \frac{\delta^2}{\delta\varphi(x)\,\delta\varphi(y)}$$

Induced product

$$\begin{array}{ccc} \mathcal{F}(\mathbb{E})[[\hbar]]^{\otimes 2} & \xrightarrow{T_E^{\otimes 2}} & \mathcal{F}(\mathbb{E})[[\hbar]]^{\otimes 2} \\ M \downarrow & \# & \downarrow \cdot_E \\ \mathcal{F}(\mathbb{E})[[\hbar]] & \xrightarrow{T_E} & \mathcal{F}(\mathbb{E})[[\hbar]] \end{array}$$

$$F \cdot_E G = \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \left\langle F^{(n)}, P^{\otimes n} G^{(n)} \right\rangle$$

$$\boxed{\mathrm{WF}(P) = \mathrm{WF}(\delta)} \quad \Rightarrow \quad F \cdot_E G \text{ ill-defined if } \mathrm{supp}(F) \cap \mathrm{supp}(G) \neq \emptyset$$

Partial Algebra of Functionals

The Partial Algebra

Lemma (Partial Algebra of Functionals)

$(\mathcal{F}(\mathbb{E})[[\hbar]], \cdot_E)$ *is an associative, commutative partial algebra, i.e.*

- $F \cdot_E G$ *is well-defined if* $\mathrm{supp}(F) \cap \mathrm{supp}(G) = \emptyset$,
- *the product is associative and commutative, if defined.*

Sketch of Proof

- Well-definedness already discussed.
- Commutativity readily follows from symmetry of $P(x - y)$.
- Associativity. $F, G, H \in \mathcal{F}(\mathbb{E})$ with pairwise disjoint supports. Calculate:

$$F \cdot_E (G \cdot_E H) = (F \cdot_E G) \cdot_E H \,.$$

∎

Euclidean Epstein-Glaser recursion
Euclidean Causality

- Associativity of $\cdot_E$ gives sense to notion of n-fold products

$$E_n(F_1 \otimes \cdots \otimes F_n) := F_1 \cdot_E \cdots \cdot_E F_n\,, \quad \forall i,j : \operatorname{supp}(F_i) \cap \operatorname{supp}(F_j) = \emptyset$$

Condition (Euclidean Causality)

Let $I \subset \{1, \ldots, k\}$ with complement $I^c \neq \emptyset$, $F_1, \ldots, F_k \in \mathcal{F}_{\mathrm{loc}}(\mathbb{E})$. If then

$$\forall i \in I,\, j \in I^c : \operatorname{supp}(F_i) \cap \operatorname{supp}(F_j) = \emptyset$$

it should follow that

$$E_k(F_1 \otimes \cdots \otimes F_k) = E_{|I|}(\bigotimes_{i \in I} F_i) \cdot_E E_{|I^c|}(\bigotimes_{j \in I^c} F_j)$$

Euclidean Epstein-Glaser recursion

Induction procedure up to thin diagonal

Induction basis For $F, G \in \mathcal{F}_{\mathrm{loc}}(\mathbb{E})[[\hbar]]$ let

$$E_0(F) = \mathbb{1}\,, \quad E_1(F) = F\,, \quad E_2(F \otimes G) = F \cdot_E G\,.$$

Induction hypothesis Let $\forall k < n$ the maps E_k

- be well-defined on the whole $\mathbb{E}^k$
- be symmetric
- fulfill Euclidean causality.

Lemma

Let E_k fulfill the induction hypothesis $\forall k < n$, then the n-fold time-ordered product E_n is uniquely defined for all functionals $\sum F_1 \otimes \cdots \otimes F_n \in \mathcal{F}(\mathbb{E})[[\hbar]]$ with

$$\mathrm{supp}\left(\sum F_1 \otimes \cdots \otimes F_n\right) \cap \mathrm{Diag}(\mathbb{E}^n) = \emptyset\,.$$

- Apply theorem on the extension of distributions.

Euclidean Epstein-Glaser recursion

Induction procedure up to thin diagonal

Sketch of Proof (follows [Brunetti, Fredenhagen 2000])

- Define cover $\{U_I\}_{I\in\mathcal{I}}$ of $\mathbb{E}^n\backslash\mathrm{Diag}(\mathbb{E}^n)$ consisting of open sets

$$U_I := \{(e_1,\ldots,e_n)\in\mathbb{E}^n\backslash\mathrm{Diag}(\mathbb{E}^n):\ e_i\neq e_j\,\forall i\in I,\, j\in I^c\}$$

- On U_I define

$$E_n^I(F_1\otimes\cdots\otimes F_n) := E_{|I|}(\bigotimes_{i\in I}F_i)\cdot_E E_{|I^c|}(\bigotimes_{j\in I^c}F_j)\,.$$

They have the sheaf property: $E_n^I\big|_{U_I\cap U_J} = E_n^J\big|_{U_I\cap U_J}$

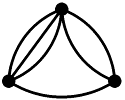
- Take partition of unity $\{\alpha_I\}_{I\in\mathcal{I}}$ subordinate to $\{U_I\}_{I\in\mathcal{I}}$ and set

$$E_n(F_1\otimes\cdots\otimes F_n) := \sum_{I\in\mathcal{I}}\left(\alpha_I E_n^I\right)(F_1\otimes\cdots\otimes F_n)$$

■

Extension to the whole space

Example



$$F(\varphi) = \frac{1}{5!}\int \varphi(x)^5 f(x) dx$$

$$G(\varphi) = \frac{1}{4!}\int \varphi(x)^4 g(x) dx$$

$$H(\varphi) = \frac{1}{3!}\int \varphi(x)^3 h(x) dx$$

- The corresponding unrenormalized amplitude is given by:

$$\int_{\mathbb{E}^3} dx\, dy\, dz\, (P(x-y))^3\, (P(x-z))^2\, P(y-z)\, f(x)\, g(y)\, h(z)$$

You are kindly invited to focus your attention to the blackboard.

Summary and Outlook

- Epstein-Glaser renormalization can be performed on Euclidean space, without the $\star_\hbar$-product structure.
- Leads to a purely algebraic construction of the Schwinger functions of EQFT.

Open Questions

- Can the renormalized time ordered product $\cdot_{\mathcal{T}^{\mathrm{ren}}}$ be defined directly as a full product on the algebra $\mathcal{F}(\mathbb{M})[[\hbar]]$?
  - properties? particularly: associativity.
- Relation of Epstein-Glaser to other schemes, to the BPHZ scheme in particular.
  - combinatorics well understood.
  - BPHZ seemingly subtracts more terms.
  - checked in examples: additional subtractions cancel among each other. (independently: [Scheck, Häußling, Falk 2009]) ...to be proven in general.