Connes-Kreimer-Hopfalgebra and Renormalization

(CMP seminar, 18.12.2008)

The aim of this talk is to make the connection between the first talk of this series on Hopf algebras and the latest ones on Renormalization. To be more precise, what I want to present to you is how the combinatorics of BPHZ renormalization can be nicely expressed/encoded in the structure of a Hopf algebra, the so-called Connes-Kreimer or BPHZ-Hopf algebra of Feynman graphs.

Graphs and Subgraphs

- Given a (renormalizable) quantum field theory $\mathcal{T}$, we dispose of a Lagrangian $\mathcal{L}$, e.g.

$$\mathcal{L}(\phi, \partial\phi) = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{1}{3!}g\phi^3 .$$

- Let $\mathcal{T}$ be the set of field monomials in the Lagrangian $\mathcal{L}$, in our example:

$$\mathcal{T} := \left\{ \frac{1}{2}(\partial\phi)^2, \frac{1}{2}m^2\phi^2, \frac{1}{3!}g\phi^3 \right\}$$

Definition: A Feynmangraph Γ is given by finite sets

- $\Gamma^{(0)}$: set of vertices
- $\Gamma^{(1)}$: set of oriented edges ("lines")

and maps, $j \in \{0,1\}$

- $\partial_j : \Gamma^{(1)} \longrightarrow \Gamma^{(0)} \cup \overbrace{\{1, \dots, N\}}^{\text{"external vertices"}}$

where for a given edge $e \in \Gamma^{(1)}$, $\partial_0(e)$ denotes the source and $\partial_1(e)$ the range of e.

- $\iota : \Gamma^{(0)} \longrightarrow \mathcal{T}$

assigning a term in the Lagrangian $\mathcal{L}$ to each vertex. It is required that

$$\deg(\iota(v)) = |\partial_0^{-1}(v)| + |\partial_1^{-1}(v)| \quad \forall v \in \Gamma^{(0)}$$

"deg" denotes the degree of the monomial in $\mathcal{T}$.

The set of Feynman graphs defined in this way we denote by Graph($\mathcal{T}$)

- Observe that there are two monomials of degree 2 and one of 3^{rd} degree in $\mathcal{J}$. Hence it is necessary to distinguish two types of vertices of valence 2 in the graphs for the map $\iota : \Gamma^{(0)} \to \mathcal{J}$ to be well-defined.

Definition (Graph algebra)

As an algebra $\mathcal{H}(\mathcal{T})$ is the free, commutative algebra over $\mathbb{C}$ generated by pairs (Γ, w), where

- $\Gamma \in \mathrm{Graph}(\mathcal{T})$ is a 1PI graph.
- $w \in \mathcal{J}$ is a monomial with degree equal to the number of external lines of Γ.

That is, we have

- a multiplication defined by

$$\begin{aligned} m: \mathcal{H} \otimes \mathcal{H} &\longrightarrow \mathcal{H} \\ \Gamma_1 \otimes \Gamma_2 &\longmapsto \Gamma_1 \dot{\cup} \Gamma_2 \end{aligned}$$

disjoint union.

- a unity (obviously: the empty set)

$$\begin{aligned} e: \mathbb{C} &\longrightarrow \mathcal{H} \\ \lambda &\longmapsto \lambda \mathbb{1} \quad , \mathbb{1} = \emptyset . \end{aligned}$$

The nontrivial part of the CK-Hopf algebra to be defined is of course the coalgebra structure. This is also what encodes the BPHZ procedure. To define it, we need the notion of a subgraph.

Definition (Subgraph)

Let $\Gamma \in \mathrm{Graph}(\mathcal{T})$. A graph $\gamma \in \mathrm{Graph}(\mathcal{T})$, $\gamma \subset \Gamma$ is called a subgraph of Γ, if

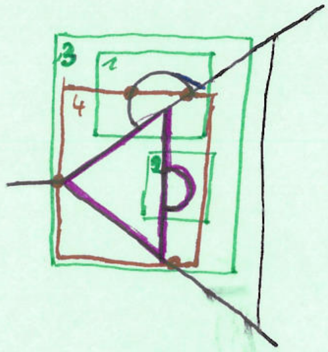
- γ is the disjoint union of 1PI graphs γ_i
- Denoting by Γ/γ the graph obtained by contracting each component of γ to a single vertex, we require:

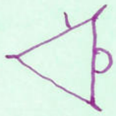
$$\Gamma/\gamma \in \mathrm{Graph}(\mathcal{T}).$$

Technical details:

- There should be a map
$$\chi : \{\gamma_i\} \longrightarrow \mathcal{J}$$
- The set of internal lines fulfills $(\gamma_i)^{(1)}_{int} \subset \Gamma^{(1)}_{int} \;\forall i$
- The set of vertices of the component γ_i is given by
$$(\gamma_i)^{(0)} = \Gamma^{(0)} \cap \gamma_i$$
- The set of external lines at a vertex $v \in (\gamma_i)^{(0)}$ is given by
$$(\gamma_i)^{(1)}_{ext} = \dot{\bigcup_{j \in \{0,1\}}} \partial_j^{-1}(v) \cap \left(\Gamma^{(1)} \setminus (\gamma_i)^{(1)}_{int}\right).$$

Example



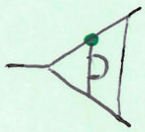
not allowed: 

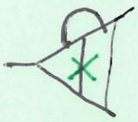
$\Gamma/\gamma_4 =$ 

4-vertex

but no monomial of degree 4 in $\mathcal{J}$!

allowed:

$\Gamma/\gamma_1 =$ 

$\Gamma/\gamma_2 =$ 

$\Gamma/\gamma_3 =$ 

Additionally ~~Actually~~ γ_1 and γ_2 are disjoint and their product can be seen as ~~the product of~~ one single subgraph: $\gamma = \gamma_1 \dot{\cup} \gamma_2$

$\Gamma/\gamma =$ 

- The set of subgraphs of a given graph Γ we denote by $V(\Gamma)$.

Definition (Graph coalgebra)

On $\mathcal{H}(\mathcal{T})$ we define

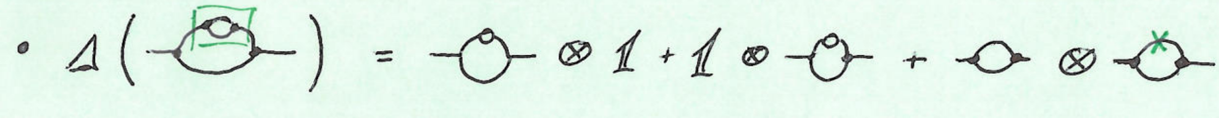
- a **counit**

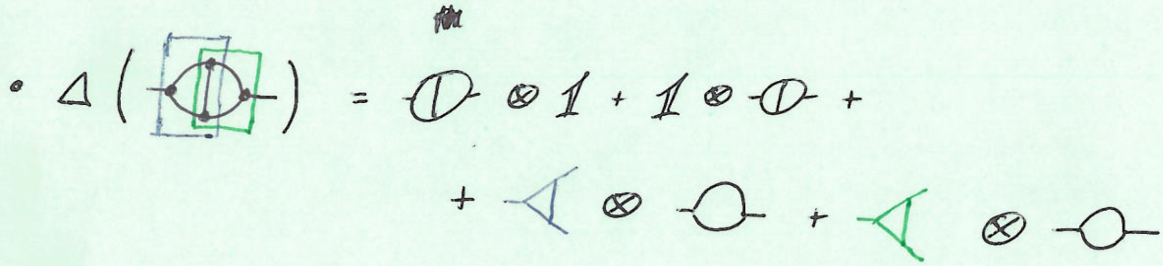
$$\varepsilon: \mathcal{H} \longrightarrow \mathbb{C}$$
$$\Gamma \longmapsto \varepsilon(\Gamma) = \begin{cases} 1 & \text{if } \Gamma = \mathbb{1} \\ 0 & \text{else.} \end{cases}$$

- a **coproduct.** (encodes BPHZ - nontrivial at last)

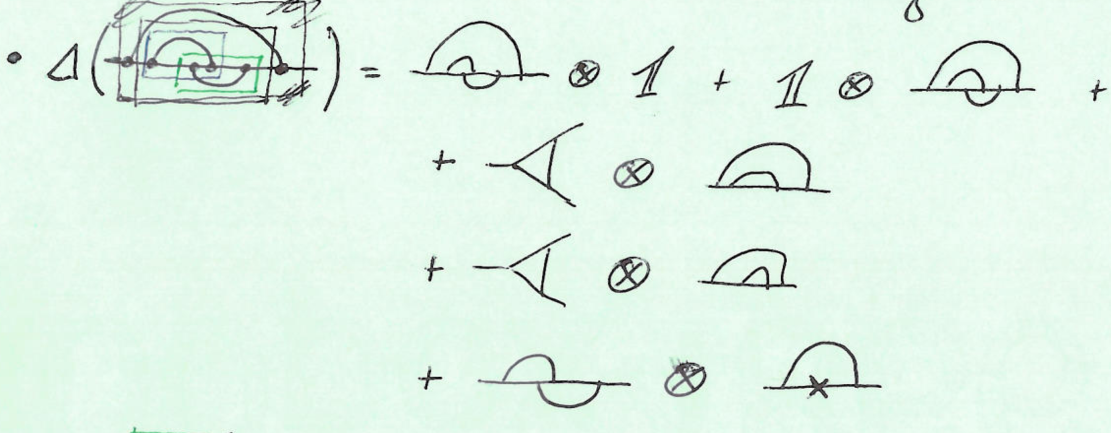
$$\Delta: \mathcal{H} \longrightarrow \mathcal{H} \otimes \mathcal{H}$$
$$\Gamma \longmapsto \Delta(\Gamma) := \mathbb{1} \otimes \Gamma + \Gamma \otimes \mathbb{1} + \sum_{\gamma \in V(\Gamma)} \gamma \otimes \Gamma/\gamma .$$

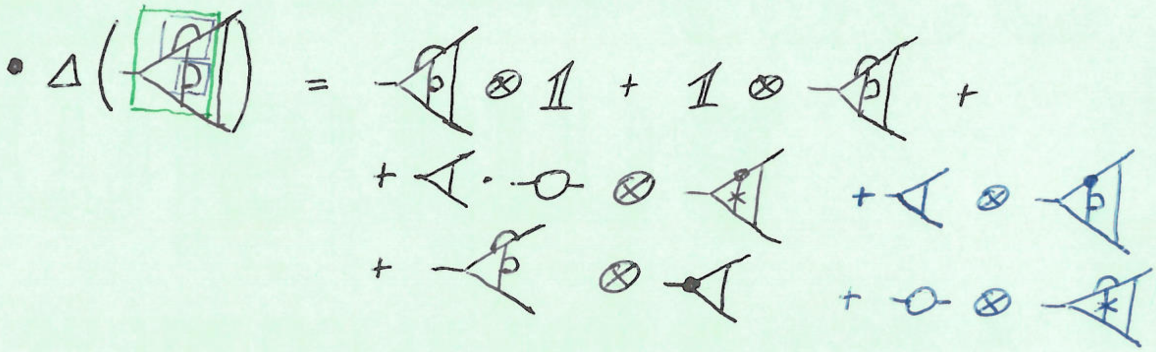
Examples





Observe that this is the same graph as , given as an example for overlapping divergencies in the last talk by K. Fredenhagen.





- It can be shown, that $\mathcal{H}(\mathcal{T})$ with the above given structures really is a ~~[illegible]~~ commutative, non-cocommutative **bialgebra**.
  - → coassociativity ~~[Kreimer2000]~~ [ConnesKreimer2000] Part I: Renorm and the Riemann-Hilbert
  - → coproduct and counit are algebra-homomorphisms.
- What is left to be defined is the **antipode**, which can be constructed inductively due to the fact that $\mathcal{H}(\mathcal{T})$ is **graded**.

Grading

- $\mathcal{H}(\mathcal{T})$ is graded as a bialgebra, i.e.

$$\mathcal{H} = \bigoplus_{n \in \mathbb{N} \cup \{0\}} \mathcal{H}^{(n)},$$

but the grading is not unique.

- grading by <u>number of edges</u>

$$E : \mathrm{Graph}(\mathcal{T}) \longrightarrow \mathbb{N} \cup \{0\}$$
$$\Gamma \longmapsto E(\Gamma) := |\Gamma^{(1)}_{int}|$$

- grading by <u>number of vertices</u>

$$V : \Gamma \longmapsto V(\Gamma) := |\Gamma^{(0)}| - 1$$

(ill-defined for empty graph) ⇒ define for connected graphs only

- grading by <u>number of loops</u>

$$\ell : \Gamma \longmapsto \ell(\Gamma) := E(\Gamma) - V(\Gamma)$$

- Observe that for each grading $\delta \in \{E, V, \ell\}$:

$$\delta(\gamma) + \delta(\Gamma/\gamma) = \delta(\Gamma)$$

has to be fulfilled to be graded as bialgebra

  - for E this is obvious, since $(\gamma)^{(1)}_{int} \subset \Gamma^{(1)}_{int}$ is exactly the set of edges contracted to one point in Γ/γ.

- for V observe that each component γ_i of γ with $V(\gamma_i)+1$ vertices is replaced by one vertex in Γ/γ. Hence

$$\sum_{\gamma_i \subset \gamma} V(\gamma_i) + V(\Gamma/\gamma) = V(\Gamma)$$

<u>Calculation:</u> Let γ have n components $\gamma_1, \dots, \gamma_n$. Then:

$$\sum_{i=1}^{n} V(\gamma_i) + V(\Gamma/\gamma) = \sum_{i=1}^{n} [|(\gamma_i)^{(0)}| - 1] + \underbrace{|(\Gamma/\gamma)^{(0)}|}_{\overbrace{|\Gamma^{(0)}| - |\gamma^{(0)}| + n}} - 1$$

$$= \sum_{i=1}^{n} |(\gamma_i)^{(0)}| - n + |\Gamma^{(0)}| - |\gamma^{(0)}| + n - 1$$

$$= |\Gamma^{(0)}| - 1 = V(\Gamma).$$

- and ℓ is a grading since it is the difference of two other gradings.

- The bialgebra $\mathcal{H}$ is <u>$\mathbb{N}$-graded-connected</u> with respect to the loopgrading ℓ and the edgegrading E. That is

$$\mathcal{H}^{(0)} = \mathbb{C} \cdot \mathbb{1}. \left(\Leftrightarrow \ell(\Gamma), E(\Gamma) \geq 0 \right)$$

- This fact makes the existence of an antipode automatic.

The Antipode

- for abbrevation we write the coproduct in some kind of "Sweedler notation" where we omit the sum:

$$\Delta(X) = \mathbb{1} \otimes X + X \otimes \mathbb{1} + X' \otimes X''.$$

we also write X instead of Γ since this is a general construction for $\mathbb{N}$-graded-connected Hopf algebras.

- $\mathbb{N}$-graded-connectedness tells us that

$$\delta(X) > \delta(X'), \delta(X'') > 0,$$

since the only element of the bialgebra for which $\delta = 0$ is the unit $\mathbb{1}$, but this does not occur in the non-trivial part of the coproduct.

- The antipode $S: \mathcal{H} \to \mathcal{H}$ is defined to be the inverse of $id_{\mathcal{H}}: \mathcal{H} \to \mathcal{H}$ in the group of Endomorphisms $(End(\mathcal{H}), \star)$:

$$S \star id_{\mathcal{H}} = m \circ (S \otimes id_{\mathcal{H}}) \circ \Delta = e \circ \varepsilon$$

- Now S is inductively constructed by setting $S(\mathbb{1}) = \mathbb{1}$ and requiring

$$S(X) = -X - S(X')X''$$

"Proof:" For nonempty graphs, i.e. X for which $\varepsilon(X) = 0$:

$$\begin{aligned}(S \star id_{\mathcal{H}})(X) &= m \circ (S \otimes id_{\mathcal{H}})(X \otimes \mathbb{1} + \mathbb{1} \otimes X + X' \otimes X'') \\ &= S(X) + X + S(X')X'' \\ &= -X - S(X')X'' + X + S(X')X'' = 0 \\ &\cancel{= \phi = \mathbb{1} = e \circ \varepsilon(X)} \quad = e \circ \varepsilon(X)\end{aligned}$$

- This is already the essential combinatorics of the BPHZ-procedure, ("subgraphs" are removed first").
- So we have seen that the antipode gives us "something finite" by removing "subgraphs first" in an inductive procedure.
- What we have to do now is to tell it exactly what to remove (and what not).

Renormalization

- **Feynman rules** in this formalism are just **algebra homomorphisms**

$$\phi : \mathcal{H} \longrightarrow \mathcal{A}$$
$$\Gamma \longmapsto \phi(\Gamma)$$

'unrenormalized amplitude corresponding to Γ.

- In general $\phi(\Gamma)$ is a divergent integral and one uses a regularization prescription to exhibit the ~~divergent~~ singularity structure of $\phi(\Gamma)$.
- In Connes-Kreimer-Theory of renormalization dimensional regularization is used, due to its nice properties, when it comes to gauge symmetries

Example

$$\phi(-\!\bigcirc\!-) = C \int d^d q \frac{1}{(p-q)^2 - m^2} \frac{1}{q^2 - m^2}$$

- Schwinger parametrization

$$\frac{1}{q^2 - m^2} = \int_0^\infty da \, \exp\left[-a(q^2 - m^2)\right]$$

- Completing the square in q
- Using generalized Gauß-Integral:

$$\int d^d q \, \exp(-q^2) = i \pi^{\frac{d}{2}}$$

- Reparametrization of the remaining integral

$$z = a + b \qquad x = \frac{a}{a+b}$$

yields Γ-function in z-integral

$$\phi(-\!\bigcirc\!-) = C \, \Gamma(2 - \tfrac{d}{2}) \int_0^1 dx \left[m^2 - p^2 x(x-1)\right]^{\frac{d}{2} - 2}$$

- ~~$\Gamma(2-\frac{d}{2})$ hat Pole bei $d = 4, 6, 8, \dots$~~
- $\Gamma(2 - \frac{d}{2})$ has poles at $d = 4, 6, 8, \dots$

- We set $\varepsilon = d-4$ and expand $\Gamma(\varepsilon)$ around $\varepsilon = 0$:

$$\Gamma\left(2-\frac{d}{2}\right) = \frac{1}{2-\frac{d}{2}}\,\Gamma\left(3-\frac{d}{2}\right)$$

$$= \frac{2}{\varepsilon} + \text{const.} + \mathcal{O}(\varepsilon)$$

- What we get is a Laurent series in the regularization parameter ε.

- As discussed by K. Fredenhagen in this seminar two weeks ago, the ~~singularity~~ simple pole can be removed by a redefinition of the mass parameter m in the original Lagrangian (mass renormalization).

- In our case this is done by adding a counterterm $\delta m^2 \sim \frac{1}{\varepsilon}$ to $\mathcal{L}$.

- This corresponds to subtracting the pole part $\sim \frac{1}{\varepsilon}$ in $\phi(-\!\bigcirc\!-)$. This is why this renormalization prescription is called minimal subtraction.

- This renormalization procedure is incorporated in the abstract setting, by saying that the Feynman rules are homomorphisms into the algebra of Laurent series, which is a direct sum in the obvious way:

$$\underbrace{A}_{} = \underbrace{A^-}_{\substack{\text{main part} \\ (\text{divergent for } \varepsilon \to 0)}} \oplus \underbrace{A^+}_{\text{finite part}} \qquad (\text{"Birkhoff sum"})$$

- The example above corresponds to setting

$$\phi^{\mathrm{ren}}(-\!\bigcirc\!-) = \phi(-\!\bigcirc\!-) - T(\phi(-\!\bigcirc\!-)),$$

where

$$T: A \longrightarrow A^-$$

is the projection to the pole part (renormalization map).

- This is the way how to extract "something sensible" from a rather simple graph , without any subdivergencies.

- The way to extract "something sensible" from a more complicated graph, now is just a matter of combining the above defined maps:

$$S_T^\phi := T \circ \phi \circ S : \mathcal{H} \longrightarrow \mathcal{A}^-,$$

~~and~~ The renormalized amplitude for a general graph now reads:

$$\phi^{ren}(\Gamma) = (S_T^\phi \star \phi)(\Gamma),$$

which is just the famous forest formula of Zimmermann, introduced already last time.

END OF TALK

- $S_T^\phi(\Gamma)$ gives the counterterms including the overall and all subdivergencies of Γ.

- To see this we expand the forest formula for a given graph Γ:

$$\begin{aligned}(S_T^\phi \star \phi)(\Gamma) &= m_A \circ (S_T^\phi \otimes \phi) \circ \Delta(\Gamma)\\ &= m_A \circ (S_T^\phi \otimes \phi) \circ \Big(\mathbb{1} \otimes \Gamma + \Gamma \otimes \mathbb{1} + \sum_{\gamma \in V(\Gamma)} \gamma \otimes \Gamma/\gamma\Big)\\ &= \phi(\Gamma) + S_T^\phi(\Gamma) + \sum_\gamma S_T^\phi(\gamma)\, \phi(\Gamma/\gamma)\\ &= \phi(\Gamma) - T\Big[\phi(\Gamma) + \sum_\gamma S_T^\phi(\gamma)\,\phi(\Gamma/\gamma)\Big]\\ &\qquad + \sum_\gamma S_T^\phi(\gamma)\,\phi(\Gamma/\gamma)\end{aligned}$$

- The first observation is that the subdivergencies are removed inductively.

- Comparing this with the original version of the forest formula of K. Fredenhagen's talk:

$$\phi^{ren}(\Gamma) = \sum_{\substack{\text{forests}\\ \mathcal{U}}} \overleftarrow{\prod_{\gamma \in \mathcal{U}}} \Big[(-t_\gamma)\,\phi(\Gamma)\Big]$$

- We see that here there is a product of renormalization maps applied to the amplitude.
- Whereas in the Hopf algebraic approach S_T^ϕ is applied to subgraphs which may also well be products of 1PI graphs.
- Hence in order to really get back Zimmermanns forest formula, and hence get local counterterms we have to ask for

$$S_T^\phi(\gamma_1\gamma_2) = S_T^\phi(\gamma_1)\, S_T^\phi(\gamma_2)$$

$$\Leftrightarrow T \circ \phi \circ S(\gamma_1\gamma_2) = \cdots$$

$$= T\left(\phi(S(\gamma_2))\,\phi(S(\gamma_1))\right)$$

- So multiplicativity would be fulfilled if we require T to be multiplicative. ?
  But: T is in general not multiplicative: !

$$\left(\frac{1}{\varepsilon}+\alpha\right)\left(\frac{1}{\varepsilon}+\beta\right) = \frac{1}{\varepsilon^2} + \frac{\alpha+\beta}{\varepsilon} + \alpha\beta$$

$$\neq \frac{1}{\varepsilon^2}$$

- Is there a weaker condition on T which also entails multiplicativity of S_T^ϕ ?
- Yes, there is. The Rota-Baxter-condition:

$$T(ab) + T(a)\,T(b) = T\left[(T(a))\,b + a\,(T(b))\right].$$

[Kreimer1999]

- Minimal subtraction fulfills Rota-Baxter:

$$T: A \to A^- \qquad A = A^+ \oplus A^-, \text{ Birkhoffsum}$$

$$T[(a^+ + a^-)(b^+ + b^-)] + T(a^+ + a^-)\,T(b^+ + b^-)$$
$$= T[a^+b^+ + a^+b^- + a^-b^+ + a^-b^-] + a^-b^-$$
$$= T[a^+b^- + a^-b^+ + a^-b^- + a^-b^-] = T[T(a)\,b + a\,T(b)]$$

done in greater detail by A. Frabetti in second talk.

- Now that we have the Connes-Kreimer Hopf algebra of Feynman graphs, we also have its "group of characters" $G^{CK}(A)$, which is a proalgebraic group since $\mathcal{H}^{CK}$ is not finitely generated.
It is the dual of $\mathcal{H}^{CK}$ an can be seen as a functor

$$G_{\mathcal{H}^{CK}} : \left\{\begin{matrix}\text{commutative}\\ \text{associative}\\ \text{algebras}\end{matrix}\right\} \longrightarrow \{\text{Groups}\}$$

$$A \longmapsto \mathrm{Hom}_{Alg}(\mathcal{H}, A)$$

as discussed in the very first talk of this seminar series.
$G^{CK}(A)$ is called the group of diffeographisms.

- Connes and Kreimer showed that there is an inclusion of the Hopf algebra $\mathcal{H}^{dif}$ (coordinate ring) associated with the group of formal diffeomorphisms.

$$\mathcal{H}^{dif} \hookrightarrow \mathcal{H}^{CK}$$

- induced by the renormalization of coupling constants. [CK 2001] : Riemann-Hilbert II

- It was, as far as I know, suggested by A. Frabetti that this inclusion may better be understood in terms of its dual map, a projection on the corresponding groups:

$$\pi : G^{CK}(A) \longrightarrow\!\!\!\!\rightarrow G^{dif}(A)$$

generalized graph exponential series $f = \sum_{\Gamma} f_\Gamma \lambda^\Gamma \longrightarrow \pi(f)(\lambda) = \sum_{n=0}^{\infty} \Big(\sum_{V(\Gamma)=n+1} f_\Gamma \Big) \lambda^{n+1}.$

Where all the coefficients of graphs having the same ~~point~~ number of vertices are mapped to just ~~the~~ one coefficient ~~of~~ ~~just one~~ in of the series expansion of a formal diffeomorphism.

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