Renormalization Hopf Algebra Emerging from Stora's Main Theorem

Dimensional Regularization in Position Space and a Forest Formula for Epstein-Glaser Renormalization

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- Vortrag zur Disputation -

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Stückelberg & Petermann ('53): *Local Freedom in S-matrix*
Popineau & Stora ('82): *Main Theorem of Renormalization*

Bogoliubov & Parasiuk ('57), Hepp ('66): *Recursive Construction*

Zimmermann ('69)
Momentum Sp, Forest Formula
Bollini & Giambiagi,
'tHooft & Veltman ('72)
DimReg+MS
Astonishing Accordance with Experiments

↓ This Talk ↓

Epstein & Glaser ('73)
Locality, Position space
Conceptually clear
Brunetti & Fredenhagen ('00)
Curved Spacetime

Kreimer & Connes ('98, '00)

Hopf algebra

Gracia-Bondía & Lazzarini, Pinter ('00)

Overview

pAQFT - Algebra of Observables

Classical Field Theory

Observables in Classical Field Theory: $f, g \in \mathscr{D}(\mathbb{M})$

$$F : \mathscr{E}(\mathbb{M}) \to \mathbb{C},\ \varphi \mapsto F(\varphi) \quad \left[= \int f(x)\,(\partial\varphi(x))^2 + g(x)\,(\varphi(x))^4\,dx \right]_{\text{e.g.}}$$

Smooth Functionals:

$$\forall n : \quad \left\langle F^{(n)}(\varphi), h^{\otimes n} \right\rangle = \frac{d^n}{d\lambda^n} F(\varphi + \lambda h)\Big|_{\lambda=0} \in \mathbb{C}, \quad h \in \mathscr{E}(\mathbb{M}).$$

Definition (Deformable Algebra)

Poisson algebra $(\mathcal{F}(\mathbb{M}), \lfloor\cdot,\cdot\rceil, \cdot)$, such that $\forall F, G \in \mathcal{F}(\mathbb{M})$:

- Pointwise Product: $(F \cdot G)(\varphi) := F(\varphi)\,G(\varphi)$.
- Poisson structure: $\lfloor F, G \rceil(\varphi) := \langle F^{(1)}, \Delta G^{(1)} \rangle(\varphi)$, where $\Delta = \Delta_{\text{ret}} - \Delta_{\text{adv}}$.
- $F^{(n)}(\varphi) \in \mathscr{E}'(\mathbb{M}^n)$ and $\left[\text{WF}(F^{(n)})\right]_{2^{\text{nd}}} \cap \left\{ \overline{\mathcal{V}^+}^n \cup \overline{\mathcal{V}^-}^n \right\} = \emptyset$.

pAQFT - Algebra of Observables

Deformation Quantization

Bidifferential operator:

$$\Gamma_H := \int dx\, dy\, H(x,y)\, \frac{\delta}{\delta\varphi(x)} \otimes \frac{\delta}{\delta\varphi(y)}\,,$$

$H = \Delta_+ + H^{\mathrm{sym}}$: Hadamard bisolution, $\Delta_+ = (i\Delta)^+$: Wightman func.
H^{sym}: smooth, symmetric, Lorentz invariant Klein-Gordon bisolution.

Deformation:

$$\begin{array}{ccc} \mathcal{F}(\mathbb{M})[[\hbar]]^{\otimes 2} & \xrightarrow{\exp(\hbar\Gamma_H)} & \mathcal{F}(\mathbb{M})[[\hbar]]^{\otimes 2} \\ & {\scriptstyle\star}\searrow \quad \swarrow{\scriptstyle\cdot} & \\ & \mathcal{F}(\mathbb{M})[[\hbar]] & \end{array}\,, \quad F \star G := \sum_{k=0}^{\infty} \frac{\hbar^k}{k!} \left\langle F^{(k)}, H^{\otimes k} G^{(k)} \right\rangle .$$

Algebra of Observables: $(\mathcal{F}(\mathbb{M})[[\hbar]]\,, \star)$, $\quad \frac{1}{\hbar}[F,G]_\star \xrightarrow{\hbar\to 0} \lfloor F, G \rceil$

pAQFT - Time-Ordered Product

Deformation

Second Order Differential Operator:

$$\Gamma'_{H_F} := \frac{1}{2} \int dx\, dy\, H_F(x,y) \frac{\delta^2}{\delta\varphi(x)\,\delta\varphi(y)}\,,$$

$H_F := \Delta_F + H^{\text{sym}}$ Feynman-like propagator,

Time-Ordered Product. Time Ordering Operator: $\mathcal{T} := \exp(\hbar \Gamma'_{H_F})$.

$$\begin{array}{ccc} \mathcal{F}(\mathbb{M})[[\hbar]]^{\otimes 2} & \xrightarrow{\mathcal{T}^{\otimes 2}} & \mathcal{F}(\mathbb{M})[[\hbar]]^{\otimes 2} \\ \downarrow \cdot & & \downarrow \cdot_{\mathcal{T}} \\ \mathcal{F}(\mathbb{M})[[\hbar]] & \xrightarrow{\mathcal{T}} & \mathcal{F}(\mathbb{M})[[\hbar]] \end{array}$$

$$, F \cdot_{\mathcal{T}} G = \mathcal{T}\left(\mathcal{T}^{-1} F \cdot \mathcal{T}^{-1} G\right) = \sum_{k=0}^{\infty} \frac{\hbar^k}{k!} \left\langle F^{(k)}, H_F^{\otimes k} G^{(k)} \right\rangle .$$

Causality: $F \cdot_{\mathcal{T}} G = F \star G$ if supp(F) later than ("$\gtrsim$") supp(G)

$\Rightarrow F \cdot_{\mathcal{T}} G$ uniquely defined iff supp$(F) \cap$ supp$(G) = \emptyset$

Partial Algebra of $\mathcal{T}$-product: $(\mathcal{F}(\mathbb{M})[[\hbar]]\,, \cdot_{\mathcal{T}})$

pAQFT - Time-Ordered Product

Graph Structure

$$\Gamma_{H_F}(F \otimes G) = \left\langle H_F, F^{(1)} \otimes G^{(1)} \right\rangle = \bullet\!\!-\!\!\bullet, \quad \delta^{\alpha} : \bigotimes_{v \in V} F_v \mapsto \bigotimes_{v \in V} F_v^{(\alpha_v)}$$

Graph expansion

$$F_1 \cdot_{\mathcal{T}} \cdots \cdot_{\mathcal{T}} F_n = \sum_{\alpha \in \mathbb{N}^n} \sum_{\Gamma \in \mathcal{G}_\alpha} \frac{\hbar^{|E(\Gamma)|}}{\mathrm{Sym}(\Gamma)} \left\langle S_\Gamma, \delta^{\alpha}(F_1 \otimes \cdots \otimes F_n) \right\rangle,$$

$\mathcal{G}_\alpha$: set of non-tadpole graphs Γ with $|V(\Gamma)| = n = \dim(\alpha)$, $|E(\Gamma)| = \frac{|\alpha|}{2}$

Definition (Local Functional)

$F \in \mathcal{F}(\mathbb{M})$ is called local ($F \in \mathcal{F}_{\mathrm{loc}}(\mathbb{M})$), if $\forall n \in \mathbb{N}$, $\forall \varphi \in \mathscr{E}(\mathbb{M})$:

- $\mathrm{supp}(F^{(n)}(\varphi)) \subset \mathrm{Diag}(\mathbb{M}^n) = \{x_1 = \cdots = x_n\}$ (thin diagonal)
- $\mathrm{WF}(F^{(n)}(\varphi)) \subset [T\mathrm{Diag}(\mathbb{M}^n)]^{\perp}$.

e.g., field polynomials.

Implication: $F^{(\alpha_v)}(\vec{x}) = \sum_{\vec{k}} f^{\alpha_v,\vec{k}}(x)\, \delta^{(\vec{k})}(\vec{r}) \in \mathscr{D}(\mathbb{M}) \otimes \mathscr{E}'_{\mathrm{Dirac}}(\mathbb{M}^{\alpha_v - 1})$

Renormalization: Find restriction S_Γ of $\bigotimes_{e \in E(\Gamma)} H_F(e)$

Renormalization Group and Main Theorem

$\mathcal{S}$-matrix

Scattering Matrix:

$$\mathcal{S} \equiv \exp_{\mathcal{T}} : \mathcal{F}_{\text{loc}}(\mathbb{M})[[\hbar]] \to \mathcal{F}(\mathbb{M})[[\hbar]] \; ,$$

Perturbative Expansion:

$$\mathcal{S}(F) = \sum_{n=0}^{\infty} \frac{1}{n!} \underbrace{\sum_{\alpha \in \mathbb{N}^n} \sum_{\Gamma \in \mathcal{G}_\alpha} \frac{\hbar^{|E(\Gamma)|}}{\text{Sym}(\Gamma)} \left\langle S_\Gamma, \delta^\alpha(F^{\otimes n}) \right\rangle}_{\mathcal{T}_n(F^{\otimes n})}$$

[C1] **Causality.** $\mathcal{S}(A+B) = \mathcal{S}(A) \star \mathcal{S}(B)$ if $\text{supp}(A) \gtrsim \text{supp}(B)$.

[C2] **Starting Element.** $\mathcal{S}(0) = 1$, $\mathcal{S}^{(1)}(0) = \text{id} : \mathcal{F}_{\text{loc}} \to \mathcal{F}_{\text{loc}}$

[C3] **φ-Locality.** $\mathcal{S}(F)(\varphi_0) = \mathcal{S}(F^{[N]}_{\varphi_0})(\varphi_0) + \mathcal{O}(\hbar^{N+1})$, $F^{[N]}_{\varphi_0}$: Taylor

[C4] **φ-Independence.** $\forall h \in \mathscr{E}(\mathbb{M}) : \left\langle \frac{\delta \mathcal{S}(F)}{\delta\varphi}, h \right\rangle = \mathcal{S}^{(1)}(0)(F) \left\langle \frac{\delta F}{\delta \varphi}, h \right\rangle$

Renormalization Group and Main Theorem

Stückelberg-Petermann Group

The Stückelberg-Petermann Renormalization Group: $\mathcal{R}$

$$Z : \mathcal{F}_{\mathrm{loc}}(\mathbb{M})[[\hbar]] \to \mathcal{F}_{\mathrm{loc}}(\mathbb{M})[[\hbar]] \,,$$

[RG1] $Z(0) = 0$

[RG2] **Starting Element.** $Z^{(1)}(0) = \mathrm{id}$.

[RG3] $Z = \mathrm{id} + \mathcal{O}(\hbar)$.

[RG4] **Locality.** Let $\mathrm{supp}(A) \cap \mathrm{supp}(C) = \emptyset$, then

$$Z(A+B+C) = Z(A+B) - Z(B) + Z(B+C)\,.$$

[RG5] **φ-Locality.** $Z(F)(\varphi_0) = Z(F^{[N]}_{\varphi_0})(\varphi_0) + \mathcal{O}(\hbar^{N+1})$.

[RG6] **φ-Independence.** $\forall \varphi \in \mathscr{E}(\mathbb{M}) : \frac{\delta Z}{\delta \varphi} = 0$.

Renormalization Group and Main Theorem

Stora's Main Theorem

Main Theorem of Perturbative Renormalization

Given two scattering matrices, $\boldsymbol{\mathcal{S}}$ and $\widehat{\boldsymbol{\mathcal{S}}}$, both fulfilling the conditions Causality, Starting Element, φ-Locality, and φ-Independence, [C1]-[C4], there is an element of the Stückelberg-Petermann Renormalization Group, $Z \in \mathcal{R}$, such that

$$\widehat{\boldsymbol{\mathcal{S}}} = \boldsymbol{\mathcal{S}} \circ Z \,.$$

Conversely, given a scattering matrix $\boldsymbol{\mathcal{S}}$ fulfilling [C1]-[C4] and a $Z \in \mathcal{R}$, then $\boldsymbol{\mathcal{S}} \circ Z$ is a scattering matrix fulfilling [C1]-[C4].

Proof: Brunetti, Dütsch, Fredenhagen 2004 / 2009.

Main Theorem in Components

Corollary (Faà di Bruno's Formula)

Let $\mathcal{S}$ fulfill [C1]-[C4], and $Z \in \mathcal{R}$, then

$$(\mathcal{S} \circ Z)^{(n)}(0) = \sum_{\mathcal{P} \in \mathrm{Part}\{1,\ldots,n\}} \mathcal{S}^{(|\mathcal{P}|)}(0) \cdot \left(\bigodot_{I \in \mathcal{P}} Z^{(|I|)}(0) \right),$$

where $\mathrm{Part}\{1,\ldots,n\}$ *is the set of all partitions of the index set* $\{1,\ldots,n\}$.

"$\odot$" denotes symmetrized tensor product: $A \odot B := \frac{1}{2}(A \otimes B + B \otimes A)$.
" $\cdot$" denotes composition of linear maps

$$\mathcal{F}_{\mathrm{loc}}(\mathbb{M})[[\hbar]]^{\otimes n} \xrightarrow{\bigodot_{I \in \mathcal{P}} Z^{(|I|)}(0)} \mathcal{F}_{\mathrm{loc}}(\mathbb{M})[[\hbar]]^{\otimes |\mathcal{P}|} \xrightarrow{\mathcal{S}^{(|\mathcal{P}|)}(0)} \mathcal{F}(\mathbb{M})[[\hbar]] .$$

Renormalization Hopf algebra

Commutative Part

Definition ($\mathfrak{H}$, commutative part)

Let $\mathfrak{H}$ be the free commutative algebra generated by differential operators

$$a_n : Z \mapsto a_n(Z) := Z^{(n)}(0)\,,$$

product: $(a_k \odot a_l)(Z) := a_k(Z) \odot a_l(Z)$; unit: $\mathbb{1}(Z) = \mathrm{id} : \mathcal{F}_{\mathrm{loc}} \to \mathcal{F}_{\mathrm{loc}}$.

Coproduct. Faà di Bruno, $\Delta a_n := \sum_{\mathcal{P}\in\mathrm{Part}\{1,\dots,n\}} a_{|\mathcal{P}|} \otimes \bigodot_{I\in\mathcal{P}} a_{|I|}$.

Grading. $\deg(a_n) := n-1$, compatible with Δ and $M_\odot$.
Starting Element condition: $a_1(Z) = Z^{(1)}(0) = \mathrm{id}$
$\Rightarrow (\mathfrak{H}, \odot, \Delta)$ is $\mathbb{N}_0$-graded connected ($a_1 = \mathbb{1}$).

Antipode. $\mathcal{A}(a_n) = -\sum_{\mathcal{P}\in\mathrm{Part}\{1,\dots,n\}\setminus\{\mathcal{P}_1\}} a_{|\mathcal{P}|} \odot \left(\bigodot_{I\in\mathcal{P}} \mathcal{A}(a_{|I|})\right)$

Renormalization Hopf algebra

Noncommutative Part

Composition product. Necessary for dualizing whole structure of $\mathcal{R}$.

$$\mathtt{C} : \mathfrak{H} \otimes \mathfrak{H} \to \mathfrak{H} \qquad , \qquad \mathtt{C}(a_k \otimes \bigodot_{i=1}^{k} a_{l_i}) = a_k \copyright \bigodot_{i=1}^{k} a_{l_i} \, ,$$

with $\left(a_k \copyright \bigodot_{i=1}^{k} a_{l_i}\right)(Z) := a_k(Z) \cdot \bigodot_{i=1}^{k} a_{l_i}(Z)$.

Compatibility. with grading, unit, and coproduct.

Convolution. $\phi, \psi \in \mathrm{Aut}(\mathfrak{H})$: $\phi \bullet_{\mathtt{C}} \psi := \mathtt{C} \circ (\phi \otimes \psi) \circ \Delta$.

Antipode.

$$\mathcal{A}_{\mathtt{C}}(a_n) = - \sum_{\mathcal{P} \in \mathrm{Part}\{1,\ldots,n\} \setminus \{\mathcal{P}_1\}} a_{|\mathcal{P}|} \copyright \bigodot_{I \in \mathcal{P}} a_{|I|} \, .$$

Algebraic Dual of Stückelberg-Petermann Group: $(\mathfrak{H}, \odot, \copyright, \Delta, \mathcal{A}_{\mathtt{C}})$

Dimensionally Regularized $\mathcal{S}$-matrix

Choosing a Feynman Propagator

Dimensionally Regularized Feynman Propagator: $d \in 2\mathbb{N}$

$$H_F^{m,\mu,\zeta}(x) = \underbrace{W_F^{m,\mu,\zeta}(x)}_{\sim K_{\frac{d+\zeta}{2}-1}} + \underbrace{B^{m,\mu,\zeta}(x)}_{\sim I_{\frac{d+\zeta}{2}-1}} \in \mathscr{D}'(\mathbb{M}), \quad \zeta \in \Omega \setminus \{0\} \subset \mathbb{C}.$$

$W_F^{m,\mu,\zeta} \xrightarrow{\zeta\to 0} \Delta_F^{m,\mu}$. $B^{m,\mu,\zeta}$: smooth, symmetric, Lorentz invariant Klein-Gordon solution.

- $H_F^{m,\mu,\zeta}$ as well as $H_F^{m,\mu,\zeta\to 0}$ depend smoothly on mass parameter m^2.
- $\forall k \in \mathbb{N}$: $\left[H_F^{m,\mu,\zeta}\right]^k \in \mathscr{D}'(\mathbb{M}\setminus\{0\})$ has *unique extension* to $\mathscr{D}'(\mathbb{M})$.
- $\forall f \in \mathscr{D}(\mathbb{M})$: $\zeta \mapsto \left\langle \left[H_F^{m,\mu,\zeta}\right]^k, f \right\rangle$ is analytic in $\Omega\setminus\{0\} \subset \mathbb{C}$, with $\mathrm{pp}\left(\left[H_F^{m,\mu,\zeta}\right]^k\right) \in \mathscr{E}'_{\mathrm{Dirac}}$ a local distribution.

Dimensionally Regularized $\mathcal{S}$-matrix

Scattering Matrix

Regularized Time-Ordering Operator

$$\boldsymbol{\mathcal{T}}^{\zeta} := \exp(\hbar \Gamma'_{H_F^{m,\mu,\zeta}}) , \qquad \Gamma'_{H_F^{m,\mu,\zeta}} := \frac{1}{2} \int dx\, dy\, H_F^{m,\mu,\zeta}(x,y) \frac{\delta^2}{\delta\varphi(x)\,\delta\varphi(y)} .$$

Regularized Time-Ordered Product

$$F \cdot_{\boldsymbol{\mathcal{T}}^{\zeta}} G := \boldsymbol{\mathcal{T}}^{\zeta} \left(\boldsymbol{\mathcal{T}}^{\zeta^{-1}} F \cdot \boldsymbol{\mathcal{T}}^{\zeta^{-1}} G \right) .$$

Regularized Scattering Matrix

$$\boldsymbol{\mathcal{S}}_{\zeta}(F) := \exp_{\boldsymbol{\mathcal{T}}^{\zeta}}(F) .$$

- satisfies conditions [C1]-[C4].
- is a *regularization outside large diagonal.*
  Let $F_1, \ldots, F_n \in \mathcal{F}_{\text{loc}}$ s.t. $\forall i \neq j$: $\text{supp}(F_i) \cap \text{supp}(F_j) = \emptyset$, then

$$\lim_{\zeta \to 0} \boldsymbol{\mathcal{S}}_{\zeta}^{(n)}(0)(F_1 \otimes \cdots \otimes F_n) = \boldsymbol{\mathcal{T}}_{\mu}^{n}(F_1 \otimes \cdots \otimes F_n) .$$

Recursion for Counterterms

There is a $Z_\zeta \in \mathcal{R}$ such that:

$$\boldsymbol{\mathcal{S}}^\mu = \lim_{\zeta \to 0} \boldsymbol{\mathcal{S}}_\zeta \circ Z_\zeta \,.$$

Writing this in components, we get

$$(\boldsymbol{\mathcal{S}}_\zeta \circ Z_\zeta)^{(n)}(0) = \sum_{\mathcal{P} \in \mathrm{Part}\{1,\ldots,n\}} \boldsymbol{\mathcal{S}}_\zeta^{(|\mathcal{P}|)}(0) \cdot \left(\bigodot_{I \in \mathcal{P}} Z_\zeta^{(|I|)}(0) \right) .$$

Corollary (Recursion for MS counterterms)

The counterterms of minimal subtraction are given by, $\mathcal{P}_1 = \{\{1,\ldots,n\}\}$,

$$Z_\zeta^{(n)}(0) = -\mathrm{pp}\left[\sum_{\mathcal{P} \in \mathrm{Part}\{1,\ldots,n\} \setminus \{\mathcal{P}_1\}} \boldsymbol{\mathcal{S}}_\zeta^{(|\mathcal{P}|)}(0) \cdot \left(\bigodot_{I \in \mathcal{P}} Z_\zeta^{(|I|)}(0) \right) \right] .$$

Interlude: Forest Formula

The Epstein-Glaser Forest Formula

MS gives *unique* extension at each step of EG, e.g., $Z_\zeta^{(2)} = -\mathrm{pp}(\boldsymbol{\mathcal{S}}_\zeta^{(2)})$.

$\Rightarrow$ Solution of Epstein-Glaser recursion is possible.

MS operator $\forall \mathcal{P} \in \mathrm{Part}\{1, \dots, n\}$:

$$-T_{\mathcal{P}}^{\mathrm{MS}} \boldsymbol{\mathcal{S}}_\zeta^{(n)} := \boldsymbol{\mathcal{S}}_\zeta^{(|\mathcal{P}|)} \cdot \left(\bigodot_{I \in \mathcal{P}} \overline{R}_{|I|} \left(\boldsymbol{\mathcal{S}}_\zeta^{(|I|)} \right) \right), \quad \overline{R}_k = \begin{cases} \mathrm{id} & \text{if } k = 1 \\ -\mathrm{pp} & \text{if } k > 1 \end{cases}$$

Forest $\mathbb{F} := \{ \cdots \geq \mathcal{P}_n \}$, tot. ord. set of partitions, $\mathcal{P}_n = \{\{1\}, \dots, \{n\}\}$

Theorem (Epstein-Glaser Forest Formula)

The finite part of the n-fold regularized time-ordered product is given by

$$\boldsymbol{\mathcal{S}}_{\zeta,\mathrm{ren}}^{(n)} := \sum_{\mathbb{F} \subset \mathrm{Part}\{1,\dots,n\}} \left(\prod_{\mathcal{P} \in \mathbb{F}}^{\geq} -T_{\mathcal{P}}^{\mathrm{MS}} \right) \boldsymbol{\mathcal{S}}_\zeta^{(n)} .$$

Renormalization Hopf algebra $(\mathfrak{H}, \odot, ©, \Delta, \mathcal{A}_{©})$

Feynman Rules and Minimal Subtraction

Z-Feynman rules. $\mathtt{feyn}_Z : a_n \mapsto a_n(Z) : \mathcal{F}_{\mathrm{loc}}^{\otimes n} \to \mathcal{F}_{\mathrm{loc}}$, linear.

- Homomorphism w.r.t. $\odot$ and ©

Regularized Feynman rules. $\mathtt{feyn}_\zeta : a_n \mapsto a_n(\mathcal{S}_\zeta) : \mathcal{F}_{\mathrm{loc}}^{\otimes n} \mapsto \mathcal{F}$, linear.

- Homomorphism w.r.t. $\odot$

Composition with Renormalization Map.

$$\left(R \circ \mathtt{feyn}_\zeta\right)(a_n) := \begin{cases} \mathrm{id} & n = 1 \\ \mathrm{pp}[a_n(\mathcal{S}_\zeta)] & n > 1\,. \end{cases}$$

- Homomorphism w.r.t. $\odot$ and © $\Rightarrow$ Rota-Baxter arg. redundant.

Counterterms. $Z_\zeta^{(n)} = \mathcal{A}_{©}^{\mathtt{feyn}_\zeta}(a_n) := R \circ \mathtt{feyn}_\zeta \circ \mathcal{A}_{©}(a_n)$.

Finitely Regularized $\mathcal{T}$-Product. $\mathcal{S}_{\zeta,\mathrm{ren}}^{(n)} = \mathtt{feyn}_\zeta \bullet_{©} \mathcal{A}_{©}^{\mathtt{feyn}_\zeta}(a_n)$.

Renormalization Hopf algebra

Graph structure

Graphs: $\Gamma \in \mathcal{G}_\alpha$: non-tadpole graph; γ_I: full vertex part of $I \subset V(\Gamma)$.

$\mathrm{Part}^c V(\Gamma)$: partitions into connected subgraphs,

i.e.,$\forall \mathcal{P} \in \mathrm{Part}^c V(\Gamma) : \forall I \in \mathcal{P} : \gamma_I$ connected.

$\Gamma/\mathcal{P}$: graph with blocks $I \in \mathcal{P}$ as vertices and as lines

all lines between different blocks $I, I' \in \mathcal{P}$.

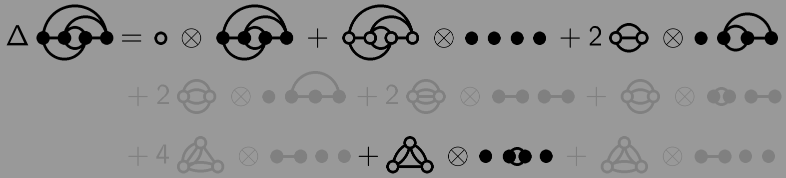
Coproduct: $$\Delta\Gamma = \sum_{\mathcal{P} \in \mathrm{Part}^c V(\Gamma)} \Gamma/\mathcal{P} \otimes \dot{\bigcup_{I \in \mathcal{P}}} \gamma_I \,.$$

Finitely Regularized Amplitude:

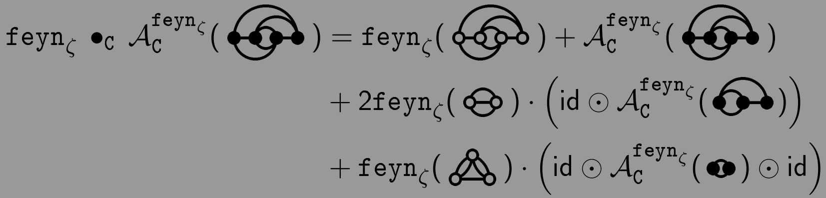
$$\left(\mathtt{feyn}_\zeta \bullet_{\mathrm{C}} \mathcal{A}_{\mathrm{C}}^{\mathtt{feyn}_\zeta}\right)(\Gamma) = \sum_{\mathcal{P} \in \mathrm{Part}^c V(\Gamma)} \mathtt{feyn}_\zeta(\Gamma/\mathcal{P}) \cdot \left(\bigodot_{I \in \mathcal{P}} \mathcal{A}_{\mathrm{C}}^{\mathtt{feyn}_\zeta}(\gamma_I)\right).$$

Example: 

Coproduct:



Finitely Regularized Amplitude:



Conclusion

- Hopf algebra structure of pQFT may be understood as the algebraic dual of the Stückelberg-Petermann renormalization group.
- To describe multiple interactions the introduction of an additional, non-commutative Hopf algebra product is necessary.
- Feynman rules and renormalization map emerge naturally and give a recursion for minimal subtraction (MS) counterterms.
- Epstein-Glaser recursion can be solved in terms of a forest formula.
- Connes-Kreimer theory of renormalization follows as a special case from Stora's main theorem.

Outlook

- Applications to the (computer-assisted) computation of higher order contributions to the perturbative expansion?
- Relation to topics in pure mathematics such as Number Theory and Noncommutative Geometry (Multiple Zeta Values, Polylogarithms, Graph Polynomials, ...) as suggested by the Connes-Kreimer framework?
- Renormalization of gauge theories also in the algebraic setting?

References

- M. Dütsch, K. Fredenhagen: *Causal Perturbation Theory in Terms of Retarded Products, and a Proof of the Action Ward Identity*, RMP, 2004
- R. Brunetti, M. Dütsch, K. Fredenhagen: *Perturbative Algebraic Quantum Field Theory and the Renormalization Groups*, ATMP, 2009
- A. Connes, D. Kreimer: *Renormalization in Quantum Field Theory and the Riemann-Hilbert Problem I+II*, CMP, 2000+2001
- H. Figueroa, J. M. Gracia-Bondía: *Combinatorial Hopf algebras in Quantum Field Theory I*, RMP, 2005

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